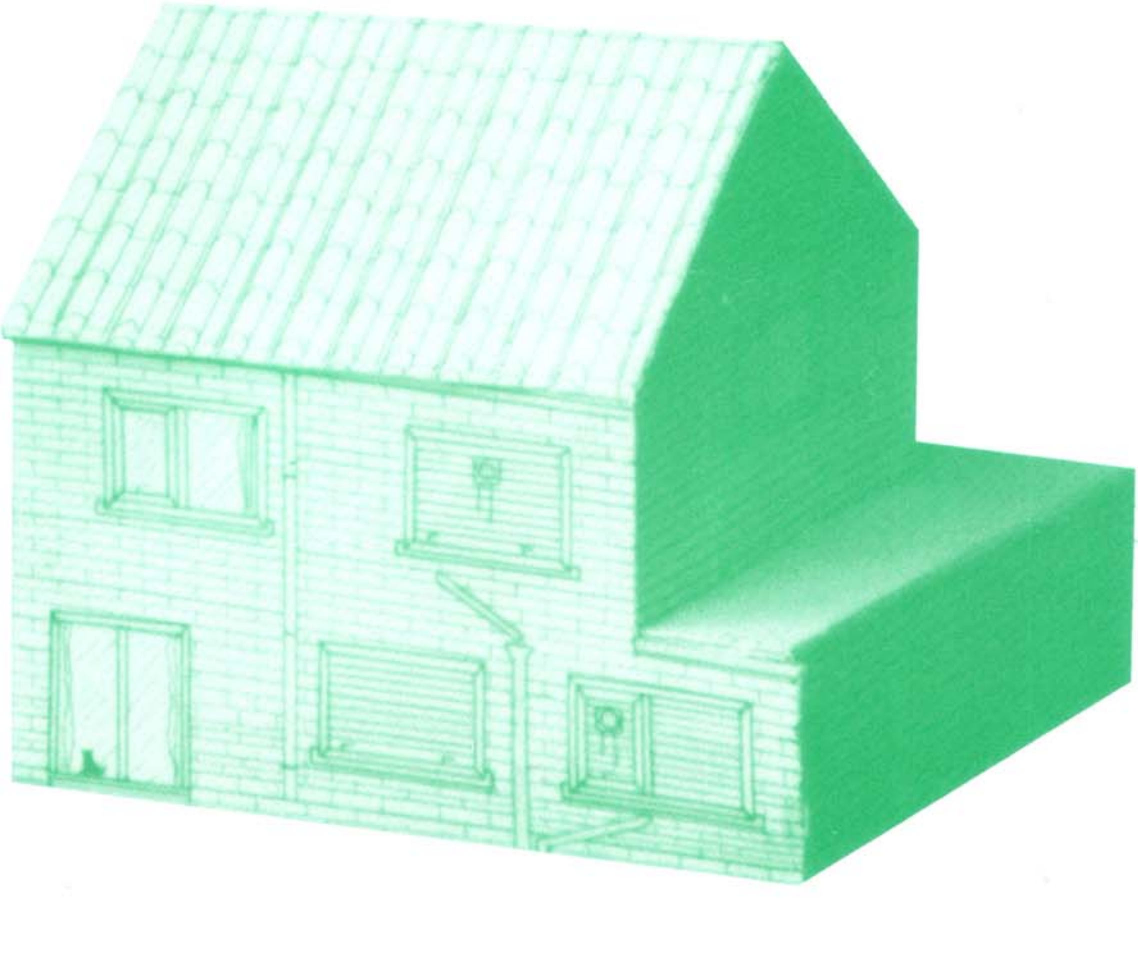
MT 126



MATHEMATICS TEACHING · MARCH 1989

March 1989

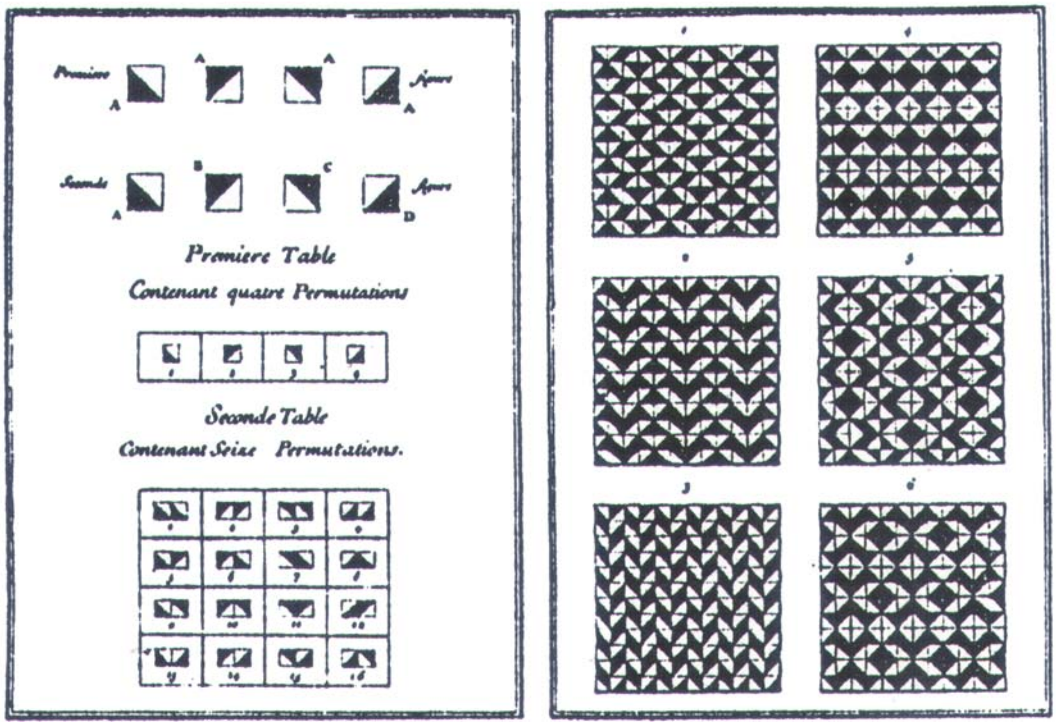
MT 126

Photograph p17 *Laurinda Brown*; illustration p23 *Kristian Russell*

MATHEMATICS AND MODERN ART: COMBINATORICS

Ulrich Grevsmühl

The systematic use of combinatorics in art and design originated in the l6th and l7th centuries when card and dice garnes were the starting points for many investigations into combinatorics and probability. In particular, Blaise Pascal, Pierre de Fermat, Gottfried Wilhelm Leibniz, Christian Huygens and Jakob Bernoulli laid the foundation for the theory which was subsequently applied to problems in economics and art. *Dominique Douat* made a systematic study of the theory of pattern, published in 1722 [1,2]. As a means of visualisation he used a square which was divided about one diagonal and then coloured using two different colours. Rotating the square by 900, the four elements A, B, C and D are formed. Gombining these into clusters gives 16 elements of twos, ie AA, AB, AG, AD, ..., 64 of threes, 256 of fours and so on. In each set the elements are taken as units for systematic generations of patterns. *(The question of how many distinct patterns are formed with 4 of these tiles making a square is discussed in Micromath, Vol 4 No.2 — Eds)*

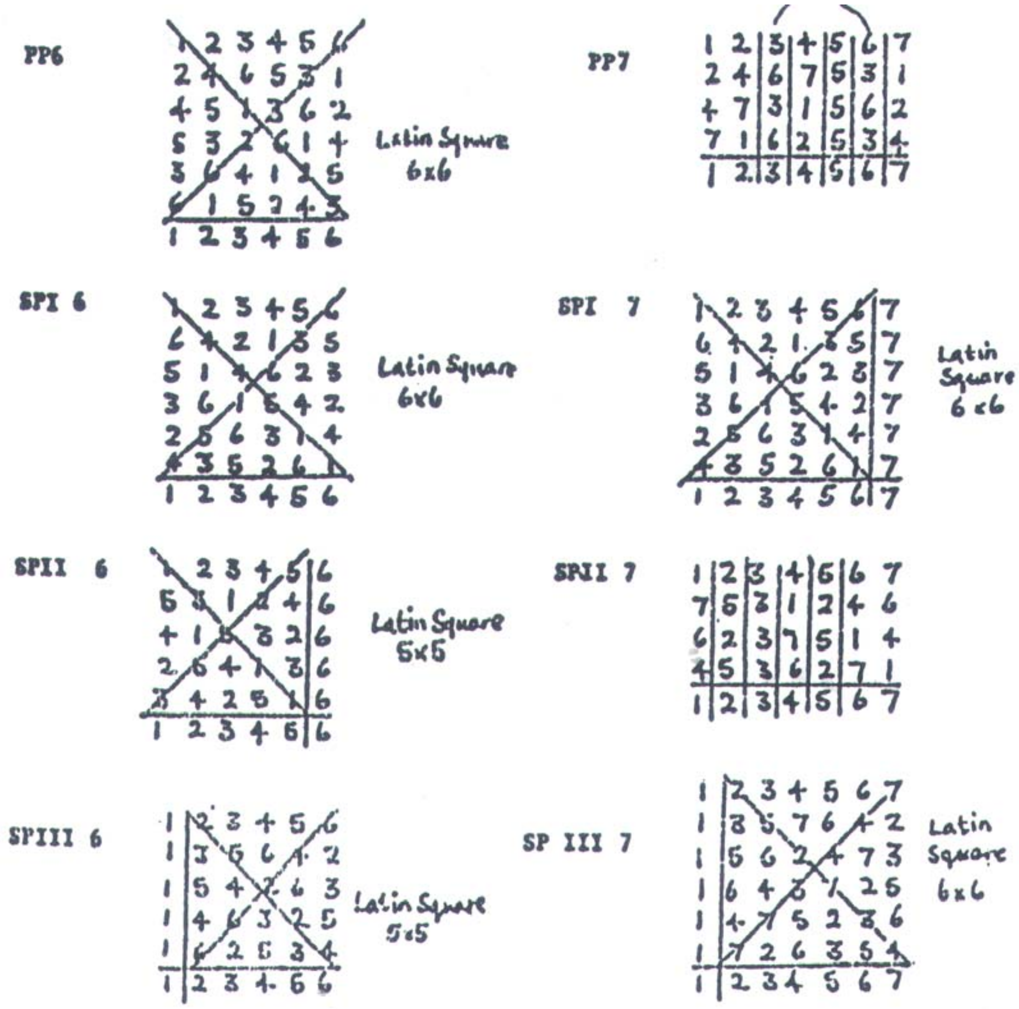


Susan Tebby

Born in Wakefield, 1944. Lives in Ullesthorpe, Lutterworth.

In contrast to Douat who generated patterns simply by means of permuting sets of elements, Susan Tebby has investigated several mathematical systems which may be used as patterns of organisation in works of art [3]. In MT124 several aspects of her work on magie squares have been discussed [4]. Here some of her findings on Pendulum and Swing Permutations are presented.

A Pendulum Permutation, (PP), is a rearrangement of the natural numbers, 1, 2, 3, ..., m in rectangular formation and is defined as an array of m columns and not more than m+ 1 rows in the following way:
If the elements in a row are given by
$e_1, e_2, e_3, \ldots, e_{2n-2}, e_{2n-1}, e_{2n}$ or $\ldots, e_{2n+1}$
where nE{l, 2, 3, ... } ie, depending if there is an even or odd number of elements, then the elements in the next row of a PP are determined by
PP: $e_2, e_4, \ldots, e_{2n}, e_{2n+1}, e_{2n-1}, \ldots, e_1$.
Other permutations may be generated by using different starting elements. Tebby distinguishes three types of Swing Permutations, (SP). Commencing a row with the last even or odd element of the proceeding row yields a SP type 1 and II, respectively:
SPI: $e_{2n}, e_{2n-2}, \ldots, e_2, e_1, e_3, \ldots, e_{2n-1}, e_{2n+1}$;
SPII: $e_{2n+1}, e_{2n-1}, \ldots, e_1, e_2, e_4, \ldots, e_{2n}$.
Starting a row with the first element of the



Examples of Penduluni and Swing Permutations of orders 6 and7.
The numbers 1, 2, ..., m appear only once in each row; the
process of permutation is complete when the original order of the first row reappears in any of the following rows; more than m+ 1 rows are needed to complete the permutation.

preceeding row, instead of the second element as for PP, one obtains a SP type III:
SPIII: $e_1, e_3, \ldots, e_{2n-1}, e_{2n+1}, e_{2n}, e_{2n-2}, \ldots, e_2$.

Pendulum Permutations were found by the artist Kenneth Martin (1905-1984) on the margin of a drawing by Paul Klee (1879-1940), but which now cannot be traced, and passed onto other artists. According to Susan Tebby no other reference to this type of permutation is known! Tebby herself discovered a mirrored version of Pendulum Permutations in a card garne outline in an early French mathematical puzzle book [5] which she then termed as Swing Permutation 1. SPII and SPIII were obtained by comparative analysis. All these permutations have been the basis for several works of an, eg Tonreihe 1-Vf (1977) but which are not shown here. The permutations display a number of interesting features and properties; obvious ones are [3]:

- constant number column, eg fifth column in PP7, last column in SPI 7
- alternating number column, eg third and sixth column in PP7
- Latin square, indicated in the examples by a diagonal cross
- multiple square, eg PP8 consists of two Latin squares of order 4 (not shown). The process of permutation takes f+ 1 rows where f is a whole factor of the number of elements in a row

Activity 1: PP and SP.
Write down all the Pendulum and Swing Permutations up to the order of 12 and investigate their properties. For a certain permutation can you find a connection between the order and its properties?

Activity 2: Computer investigation of PP and SP.
Write a computer program to calculate and print out all Pendulum and Swing Permutations up to the order of, say, 100. Can you generalise the results which you obtained in the previous activity?

RICHARD PAUL LOHSE *Born and lived in Zurich, 1902-1988*

Looking back at the development of constructive art, which is reflected in the works of the second generation of constructivists since the late thirties, exemplified in the Zurich group of Max Bill (for some of his works see [6]), Gamille Graeser, Richard Paul Lohse, Verena Loewensberg, and others, one finds that despite their individual differences these artists have put the inheritance of the Dutch De Stijl movement and the East European constructivism of the early 2Oth century on a clear rational basis. Lohse‘s aim, in particular, was to renew the movement and to give it new content by reformulating the means and methods of constructive art.
Proof of his achievement is found in his numerous impressive works which can be divided according to their orders into two large groups: modular and serial. Works with modular orders often have a starting point from which all further operations, like geometrical transformations or groupings, develop. Works with serial orders, on the other hand, show continuous series based on continuous order of colours. Combinatorical procedures are used to put the series into movement and to produce the dynamics of the picture [7, 8].
His works may be seen as models of the sociological structure of our society where the autonomous individual is represented by a colour square, clearly separated from the others, the democratic principle by the equal presence and quantity of all colours and the organic principle of growth by the growing colour ordering.
Lohse writes [9]: *The pre-existent, anonymous individual element contains within itself the combinatory principle: the group, the great number‘, the thematic series.*

Objectivisation, schematisation, stand at the beginning of a new form of visual creation. The ‘great number‘ is pictorial reality. Individual expression consists in the selection of methods, in directing the pre-set conditions.
... The combinatory principle enables formulations of orderings on a large scale ... the method represents itself and is the painting.
We shall now investigate three of his works based on the serial principle.

Activity 3: Cyclic permutations of a continuous colour series. In *30 vertical systematic colour series with red diagonals (1943-70, oil on canvas, 165 x 165cm)* (see double page spread) each column has the same thirty spectral colours of the continuous colour spectrum which ensures that the work contains the same amount of each of the colours. Cyclic permutations of the spectral colours are used to create visual patterns. Analyse the underlying structure of the

29	16																												1
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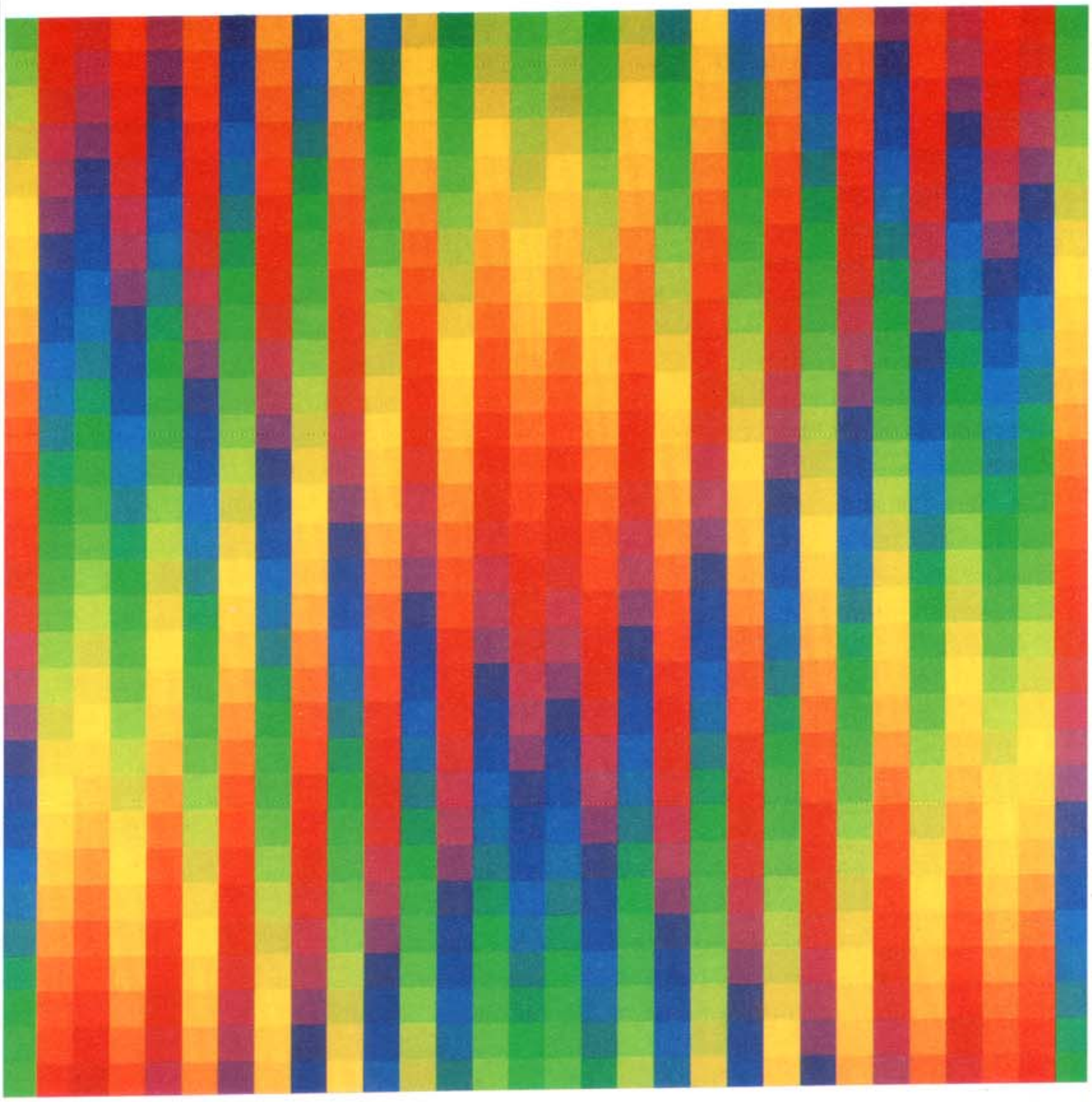
work with the help of the diagram in which each spectral colour is denoted by a number, eg 1 for a shade of green, 16 for a shade of red, and so on [7]. Explain also the pattern around the centre and at the borders of the picture.

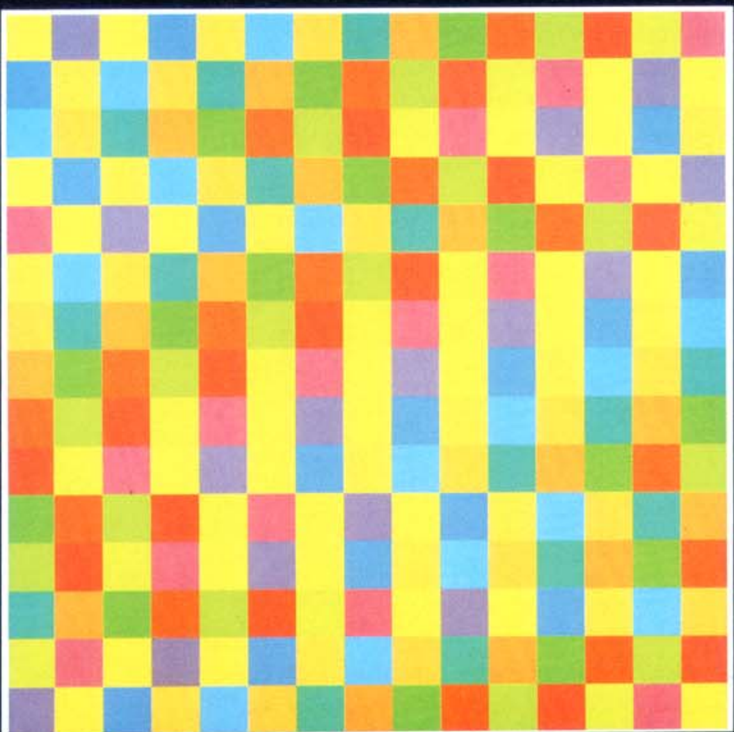
COMBINATORICS



RICHARD PAUL LOHSE

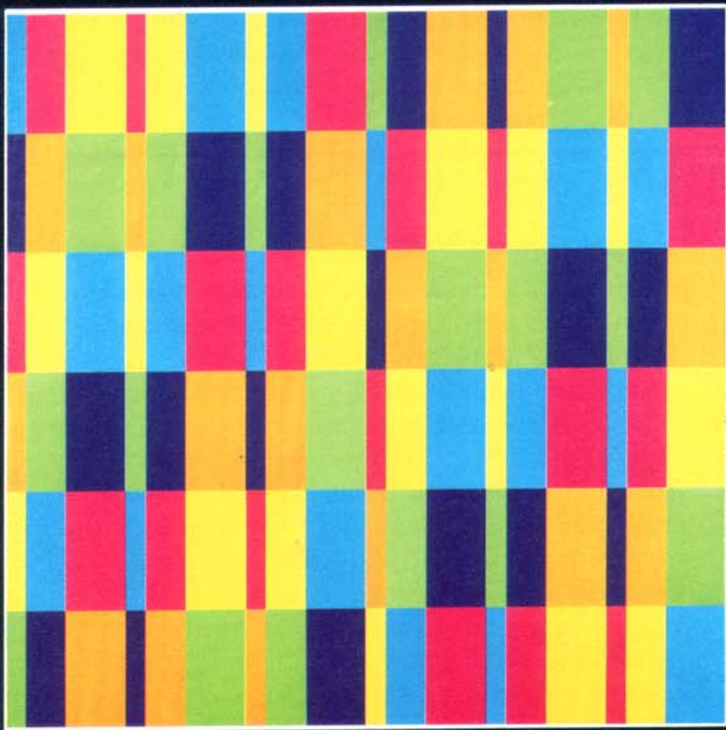
30 vertical systematic colour series with red diagonals (1943-70)

ACTIVITY 3



15 interpenetrating systematic colour series in three horizontal groups (1943-69)

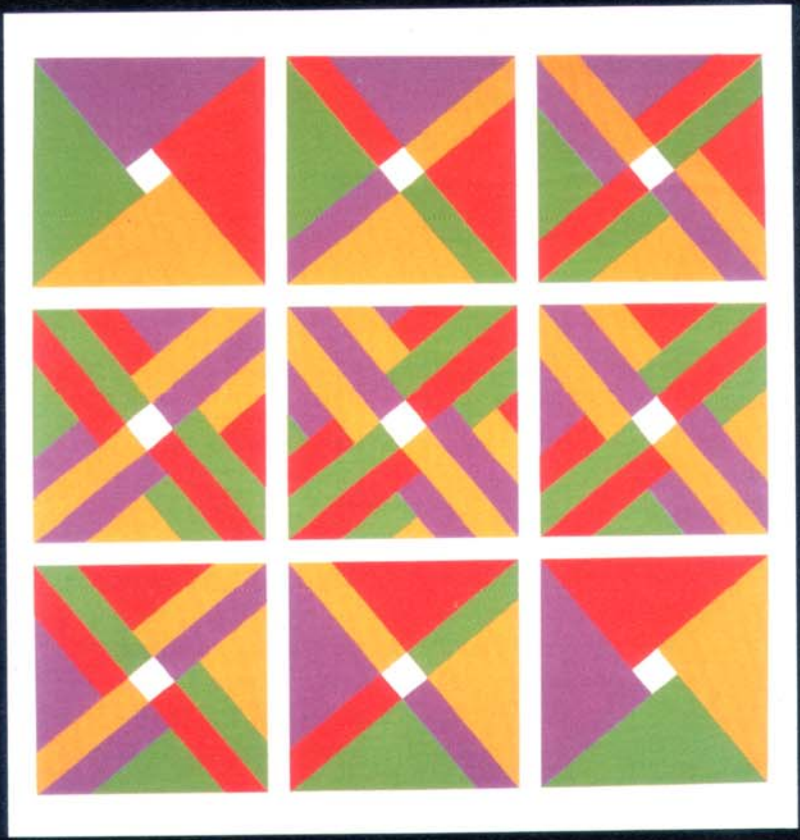
ACTIVITY 4



6 horizontal bands with 6 formally equal colour groups each (1950-51-69)

ACTIVITY 5

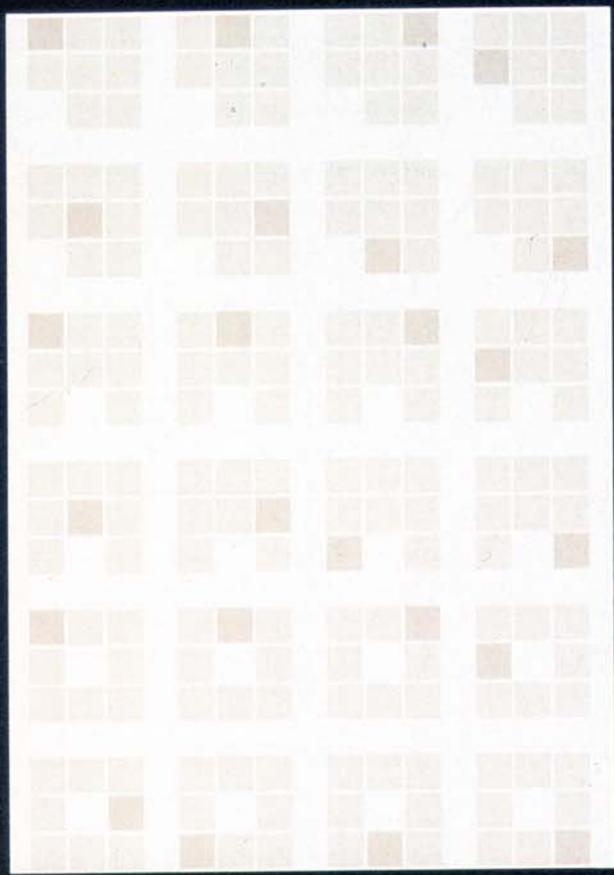
SHIZUKO YOSHIKAWA



Transformation of the four colour surface of the same area ii (1975-76)

ACTIVITY 8

WILLEM KLOPPERS

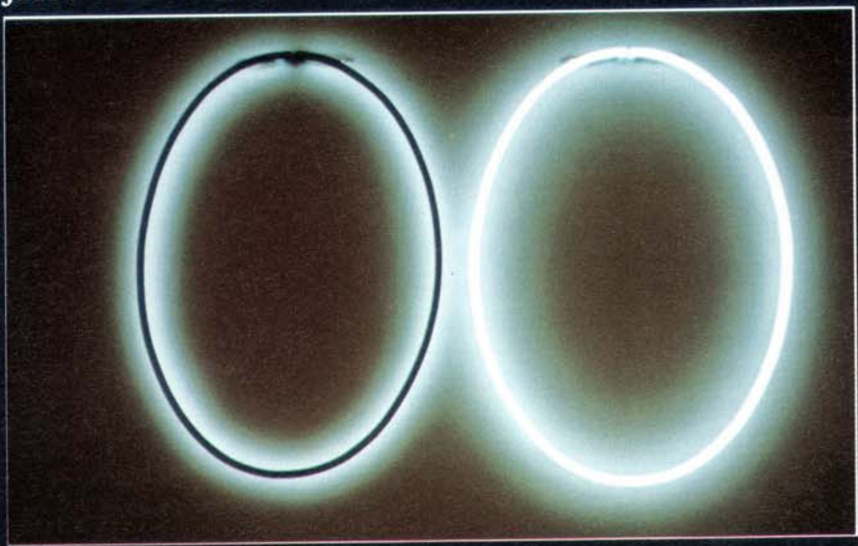


Quadratum Novem (1987)

ACTIVITY 6

JAN VAN MUNSTER

OO Light Dark (2 ovals) (1986)



ANTON STANKOWSKI

Triangle asymmetric (1954)

ACTIVITY 7

Activity 4: Cyclic permutations of a discontinuous colour series.

i) In contrast to the work in activity 3 each row of the 15 interpenetrating systematic colour series in three horizontal groups (1943-69, oil on canvas, 150 x 150cm) displays a discontinuous order of fifteen spectral colours of the continuous spectrum. In the diagram the colours are numbered according to the spectral colours of the continuous spectrum hut differently to the notation in activity 3, eg 1 for a shade of yellow, 15 for a shade of green. The structure of the work is achieved again by cyclic permutations of the first row in a 15 x 15 grid which ensures that each spectral colour occurs in each row and column only once [7]. Calculate the number of possible works that can be created with this procedure. Give an explanation of the colour structure of the 'three horizontal groups'.

15	7	14	6	13	5	12	4	11	3	10	2	9	1	8
								2		1				
3				1										
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		9	1											
	1												10	
						11					1			
							12							1
		1						13						
1				14										
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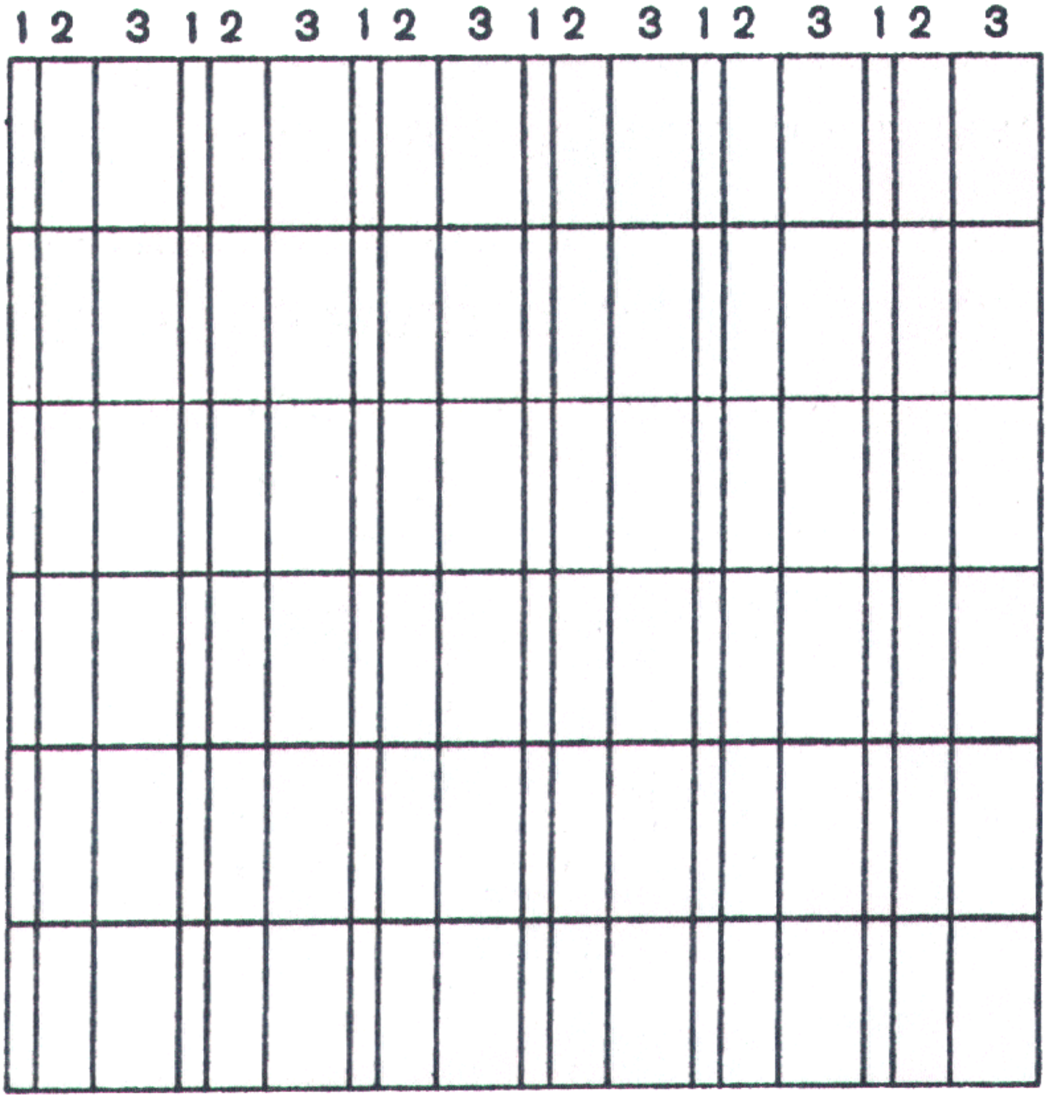
ii) Calculate the number of possible works that could be created by non-cyclic permutations of a row with fifteen elements in a 15 x 15 grid with the requirement that each row or column contains each of the fifteen spectral colours only once. Start your investigation by considering at first grids of order 3 and 4.

Activity 5: Cyclic permutations of colour groups.

i) The structure of 6 horizontal bands with 6 formally equal colour groups each (1950-51-69, oil on canvas, 126 x 126cm) contains 36 squares arranged in a 6 X 6 grid and divided into cells with areas of proportions 1:2:3. Six colours are allocated to the cells by the cyclic permutations of the discontinuous sequence 1:3:2, 2:1:3, 3:2:1 in such a way that each colour occurs with one of these proportions only once in each row and column. In the total picture each colour is equally represented by an area of six squares. In allocating the six colours to the cells, each of the three cyclic permutations is used twice, first by applying it to the primary colours blue, red, yellow. Then the secondary colours orange, green and violet are allocated in the vertical intermediate cells between red-yellow, yellow-blue and blue-red, respectively, where it is necessary to extend the structure vertically by repeating the grid [7, 8].



ii) Calculate the number of possibilities to allocate just one colour into the above grid of 108 elements by using cyclic permutations of either the continuous sequence 1:2:3 or the discontinuous sequence 1:3:2 with the requirement above that within a square the colour occurs not more than once in any of the three cells and that a row or column contains just one of the permutations.

JAN VAN MUNSTER *Born in Gorinchem, 1939. Lives in Renesse, Netherlands.*

The central themes of Jan van Munster‘s sculptures and drawings are the concepts of energy and of opposites. Many of his installations are based on the non-conventional medium of neon lights, laid out in geometrical shapes, which he uses as means of expression and realisation of his concepts. The phenomena energy in its different forms like light, sound, heat, electricity, magnetism and radioactivity is given form and tension by making perceptable to the spectator the corresponding forms of opposites:
light-dark, noise-silence, warm-cold, electrical plus minus, magnetic attraction-repulsion and also emotions like love-hate [10]. A typical example is his work 0 0 light dark (2 ovals) (1986, painted and unpainted neon, 70 x 50cm each) which is shown in the double spread. In art, opposites, like the polarisation in positive and negative or in form and anti-form, are often experienced as active dialogue which turns an interaction into a productive interplay. The combinatorical relevance of opposites is that they can be regarded as the basic elements of just two states, + or — or 0 1, switching a machine on and off, an event takes place and not, success and failure, and so on, with the one state being the negation of the other.

WILLEM KLOPPERS *Born in Den Haag, 1937. Lives in Den Haag.*

In recent years the works of Willem Kloppers have been based on mathematical orderings. Using the quadratic grid he explores visual variations of basic structures in two and three dimensions.

Activity 6: Combinatorics of a 3 x 3 grid. A square has been divided into nine smaller square elements. Find out how many possibilities there are to identify the square by marking one of its elements without taking into account rotations of the total square.

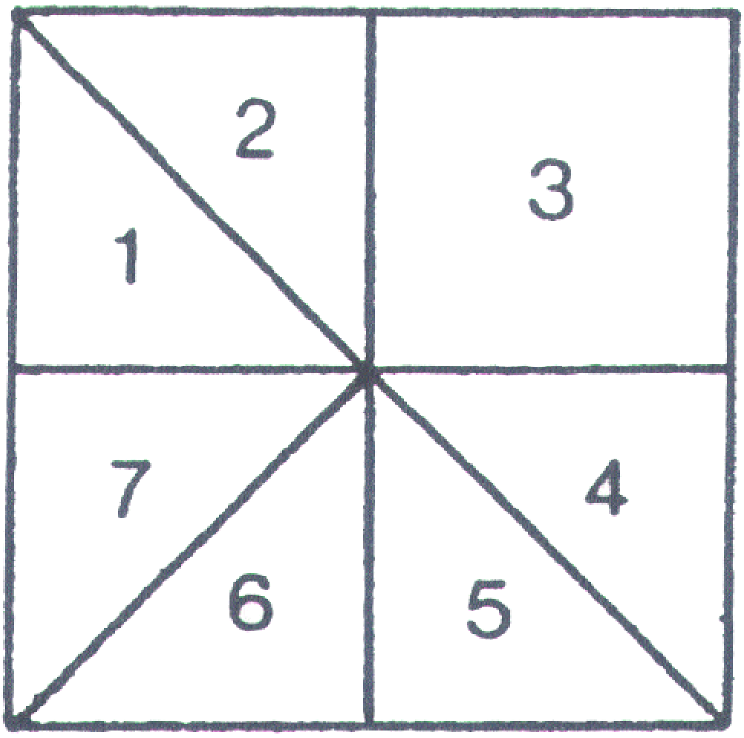
1	2	3
4	5	6
7	8	9

We shall consider complete and incomplete squares. For a complete square with nine elements there arc the three possibilities:
{1, 3, 7, 9}, {2, 4, 6, 8} and {5}. Omitting one of the nine elements there arc just 24 possibilities to mark an element within the square which are depicted in Quadratum novem (1987, silk screen print, 24.2>< 15.8cm) (see double spread). How many possibilities are there if 2, 3, ..., 8 elements are omitted?

ANTON STANKOWSKI *Born in Gelsenkirchen, 1906. Lives in Stuttgart.*

In his working life Anton Stankowski has gained a wealth of experience as a decorator and church painter, typographer, graphic designer and artist of the concrete and constructive movement. One of his special interests has been the concept of aesthetics. His fundamental thesis is: Aesthetics is order. If the order is perfect then the thing (which is represented in the picture) is destroyed!. He writes [12]:
One has to be able to realise the principle so that the spectator can use it as a dialogue with the work. To order means. leaving things out that arc not so important. By this the information is strengthened. One cannot express the essentials without making a choice about order.

Activity 7: Permutational combinations.
His work Triangle asymmetric (1954, tempera, 60>< 100cm) (see double spread) demonstrates his view on aesthetics and order by an incomplete set of a combinatorical system. The basic element of the work is a square which has been divided into seven parts. Parts 4 and 7 arc left white and there arc 12 shades of colours that are allocated to the remaining parts.



Calculate the number of possibilities in colouring the parts of the square if

i) no repetitions of colour are permitted; ii) all repetitions of colour arc permitted; iii) repetitions of colour are permitted but with the added requirement that within a square neighbouring parts with one side in common must have a different colour.

SHIZUKO YOSHJKAWA *Born in Omuta, Fukuoka, Japan, 1934. Lives in Unterengstringen, Zurich.*

Shizuko Yoshikawa's art has developed and grown in the tensional field between Eastern and Western culture. Her understanding of geometry has been greatly formed by the Japanese culture in which she grew up. In particular, the rectangular structures, found both in the wooden skeletal style of Japanese architecture and the Chinese symbols of writing, were experienced as spiritual orderings of the world. For her geometry is the purest and most elementary force of ordering and the basis of all formal and logical thought [14, 15].
In many of her works geometrical transformations arc visualised by permutational changes of colour. Yoshikawa writes [16]:
By the means of concrete art 1 have tried to visualise the general symbolic of the cycle of life, which is taken from Buddhism and in Japanese called 'groove' (= transmigration). The them 'transformation' was at the same time my own projection in order to survive spiritually on European ground. In those days 1 regarded the numerical-rational view of the world as the ideal of a new beauty. The smallest structured parts are the building elements of this world from which the manifold formations of the beauty of nature can arise.

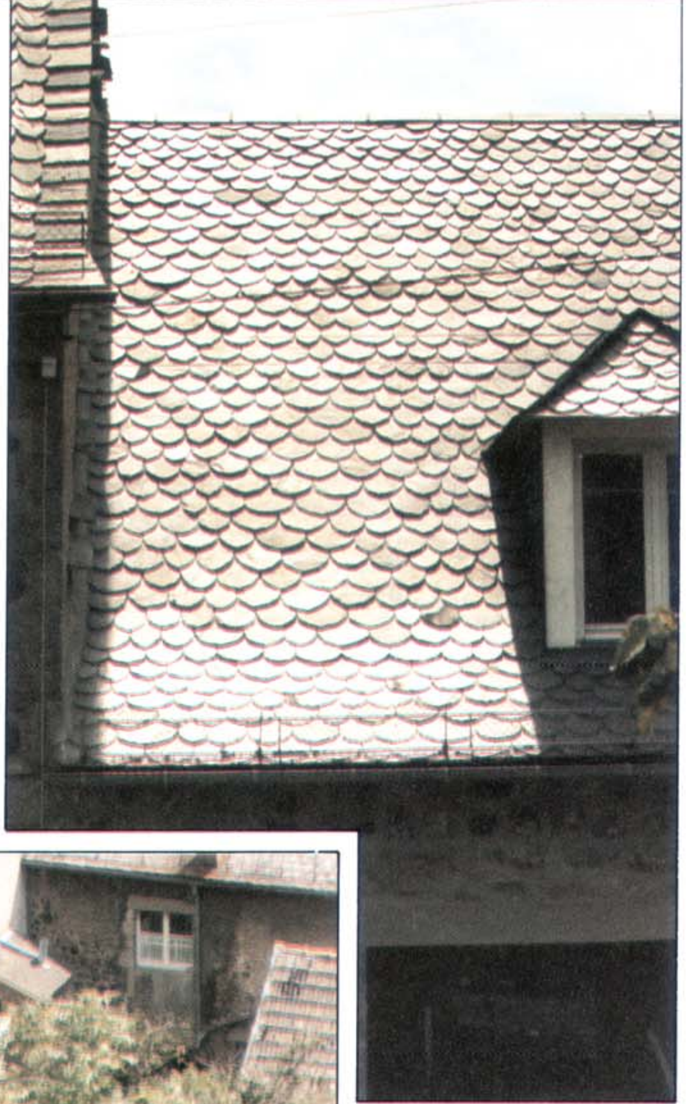
Activity 8: Permutational transformations:
Analyse the generating principle of Transformation of the four colour-surface of the same area II (1975-76, acryl on linen, single element 50 x50cm, a total of 9 elements 159>< 159cm, incl. of 4.5cm distance between the single parts) in the double spread [17]. Show for each of the nine pictures that the total area of the dissected parts carrying the same colour is unchanged. State the transformational and permutational changes that take place from one picture to the next.

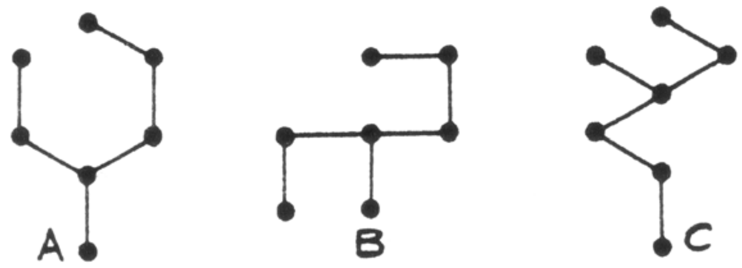
ANTHONY HILL *Born in London, 1930. Lives in London.*

Since the earliest days of his artistic career Anthony Hill has been interested in mathematical ideas. He has collaborated with several mathematicians and has actively been involved in mathematical research which resulted in a number of articles on various topics in combinatorics and graph theory [18, 19, 20]. In his works of art he puts to work abstract ideas taken from the domain of mathematics, as can be seen in the following combinatorical examples [21]. In Graph Theory the trees labeled A, B and C are by definition the same tree but differ in appearance because they are placed in grids with different symmetries. For A the grid is hexagonal, for B square and for C triangular. The trees have seven points and six lines each, are asymmetry and examples of the smallest non-trivial identity tree as its automorphism group consists only of the identity automorphism ([22] is an introduction to Graph Theory). They have been the subject of extensive studies and the impulse for several works of art.

ROOF TILINGS

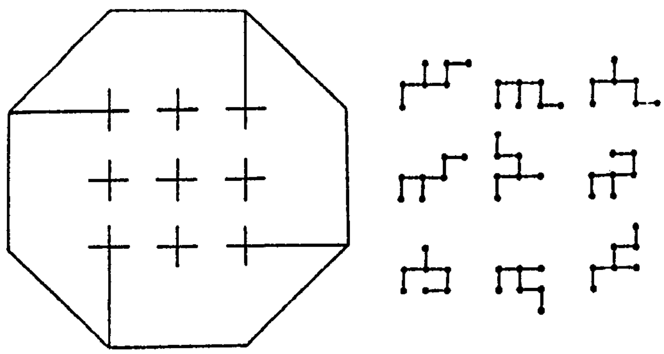
Photographs:
Ian Wood, Fylde Branch Secretary
Jenny Murray, Aldeburgh Primary School, Suffolk

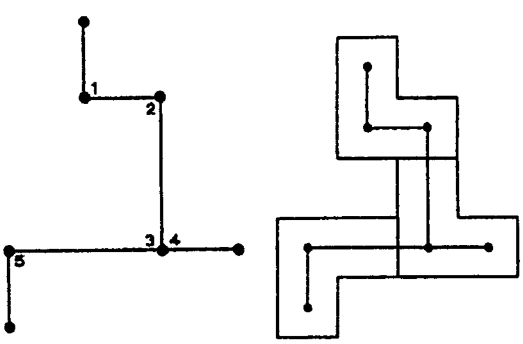




In the hexagonal grid the tree A can be laid out in eight different ways which are depicted in Parity *Study Theme 2, first version (1970-74, paint on perspex, 40 x 40")* (see p2) [18]. If for each of the eight trees its parts are rotated and permutated in all possible ways around the nodal points, ie the point from which at least three lines start, one finds that the three directions of the grid give rise to four possibilities each and a total of 96 trees with different configurations. In the case of the square and triangular grid a similar analysis leads to a total of 65 x 8 or 520 and 733 x 12 or 8796 trees of different configurations, respectively [21].
The Nine-Hommage á Khlebnikov No.2 (1976, engraved and relief laminated plastic on wood, 36x36x3/4", octagonal) [18, 21] (see p5O) is based on tree B and the square grid. From the different configurations associated with tree B those nine basic types were selected which each have five right angles and which each can be composed by three graphs of L shape. The work thus shows the nine distinct ways of embedding the smallest asymmetry tree on the orthogonal lattice such that a set of very strict parameters define the smallest family [18].
Hill's work consists of a white plastic laminated octagonal panel with engraved lines. The 27 L-shaped pieces associated with the nine trees are laid out on the panel in a 3x 3 array which contrasts with the cyclic symmetry of the engraved lattice. The L-shapes are of the same white plastic material but have black sides and give rise to different views when observed from different angles. The work is dedicated to the Soviet poet Velimir Khlebnikov (1885-1922) who played an important role in the Formalist Movement in the arts.

Pädagogische Hochschule Freiburg, West Germany





Acknowledgements
I am indebted 10 A Hill, W Kloppers, J van Munster, A Stankowski,
S Tebby and S Yoshikawa tor their collaboration. My sincerest gratitude goes to R P Lohse, who died during the final typing of this article, for his generous support and continuous interest.

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